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Addendum

Volume **20**, Number 3 (1977), in the article "The Asymptotic Cost of Lagrange Interpolatory Side Conditions" by R. K. Beatson, pp. 288–295:

Let N_n be the space of trigonometric polynomials of degree $\leqslant n$, $\{t_i\}_{i=1}^{\gamma}$ a set of γ (distinct) points in $[-\pi, \pi)$, $C(T)$ the space of continuous 2π-periodic functions, and let $A = A(f) = \{g \in C(T): g(t_i) = f(t_i); i = 1,..., \gamma\}$. Theorem 1.4 guarantees that, if $f \in C(T)$ is not a trigonometric polynomial, then $\limsup_{n\to\infty} d(f, A \cap N_n)/d(f, N_n) \leqslant 2$. Here $d(\cdot, \cdot)$ is the uniform metric. The purpose of this note is to show that the constant 2 on the right-hand side of this inequality cannot be decreased. This shows, more generally, that the constant 2 appearing in Theorems 1.4, 1.5 cannot be decreased.

LEMMA. *Let $A = \{g \in C(T): g(0) = f(0)\}$. There is a function $f \in C(T)$ such that*

$$\limsup_{n\to\infty} \frac{d(f, A \cap N_n)}{d(f, N_n)} \geqslant 2.$$

Proof. Consider the sequence of functions $\{g_i(\theta) = \cos(3^i\theta)\}_{i=1}^{\infty}$. Since

$$g_i(\theta) = (-1)^j \qquad \text{when} \quad \theta = \frac{j\pi}{3^i}; \quad j = 0, \pm 1, \pm 2,...,$$

g_i has 2.3^i extrema on $[-\pi, \pi)$. Also g_k, $k \geqslant i$ has all the extrema of g_i with the same sign as g_i. Let $\sum_{i=1}^{\infty} a_i$ be some convergent series of positive numbers, and define

$$f(\theta) = \sum_{i=1}^{\infty} a_i\, g_{2^i}(\theta).$$

Consider the residual of best uniform approximation, to f from N_n. This residual is characterized by the existence of a set of $2n + 2$ points in $[-\pi, \pi)$, its value at each such point being equal in magnitude to its norm but alternating in sign. Hence the best uniform approximation to f from $N_{3^{(2^i)}}$ is

$$h_i(\theta) = \sum_{k=1}^{i} a_k\, g_{2^k}(\theta)$$

with

$$f - h_i = r_i = \sum_{k=i+1}^{\infty} a_k\, g_{2^k}$$

0021-9045/80/080336-03$02.00/0

and

$$\|f - h_i\| = d(f, N_{3^{(2^i)}}) = \sum_{k=i+1}^{\infty} a_k ,$$

where $\|\cdot\|$ denotes the uniform norm on $[-\pi, \pi]$.

Let t_i be any function in $N_{3^{(2^i)}} \cap A$. That is, $t_i \in N_{3^{(2^i)}}$ and $t_i(0) = f(0)$. Then

$$\|f - t_i\| = \|(f - h_i) - (t_i - h_i)\| = \|r_i - p_i\|, \tag{1}$$

where $p_i = t_i - h_i$ is the perturbation of the best approximation.

The argument now proceeds using that

$$p_i(0) = r_i(0) = \|r_i\| \qquad \text{while} \quad r_i\left(\frac{\pi}{3^{(2^{i+1})}}\right) = -\|r_i\|;$$

and that the slope of $p_i(\theta)$ is related to its norm by Bernstein's inequality. We treat two cases.

Case 1. If $\|p_i\| \geqslant 3d(f, N_{3^{(2^i)}}) = 3\|r_i\|$, then $\|r_i - p_i\| \geqslant \|p_i\| - \|r_i\| \geqslant 2\|r_i\|$.

Case 2. If $\|p_i\| \leqslant 3d(f, N_{3^{(2^i)}}) = 3\|r_i\|$, then using Bernstein's inequality

$$p_i\left(\frac{\pi}{3^{(2^{i+1})}}\right) = p_i(0) + O\left(\frac{\pi}{3^{(2^{i+1})}}\|p_i'\|\right)$$

$$= \|r_i\|\left(1 + O\left(\frac{3\pi \cdot 3^{(2^i)}}{3^{(2^{i+1})}}\right)\right) = \|r_i\|(1 + o(1)),$$

and since $r_i[\pi/3^{(2^{i+1})}] = -\|r_i\|$ we find

$$\|r_i - p_i\| \geqslant \left|(r_i - p_i)\left(\frac{\pi}{3^{(2^{i+1})}}\right)\right| = 2\|r_i\|(1 + o(1)).$$

Thus from (1) and the estimates for $\|r_i - p_i\|$ above we have

$$\limsup_{i\to\infty} \left(d(f, N_{3^{(2^i)}} \cap A)/d(f, N_{3^{(2^i)}})\right) \geqslant 2.$$

Remarks. The proof of the lemma requires only that each a_i be positive and that the series $\sum_{i=1}^{\infty} a_i$ converges. Hence there is no requirement that f be "nonsmooth"; suitable choice of the a_i will in fact make f entire. Also since the function f of the lemma is even, and the constraint is at $\theta = 0$, one may use the usual change of variable $x = \cos\theta$ to obtain a result about uniform approximation by algebraic polynomials on $[-1, 1]$. This shows that the constant 2 in the theorem of S. Paszkowski ("On Approximation with Nodes," *Rozprawy Mat.* **14** (1957), 1–61), which we generalised, is best possible.

The original article contains several typographical errors. In the statement of Theorem 1.1 replace $X = [a, b]$ by $X = C[a, b]$. On page 290, line 8, replace the reference to [2, Theorem 4.1] by a reference to [2, Theorem 4.2]. In the statement of Corollary 1.6, replace the condition $f(t_i) < \|f\|$ by the condition $|f(t_i)| < \|f\|$.